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INTERIM EFFICIENT MECHANISM DESIGN WITH INTERDEPENDENT VALUATIONS

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Abstract

We consider the class of Bayesian environments with one-dimensional private signals and interdependent valuations. For these environments, we fully characterize the set of interim efficient mechanisms which satisfy interim incentive compatibility and interim individual rationality constraints. In these mechanisms, the allocation rules assign probability one to an alternative that maximizes the sum of all agents' virtual valuations that are defined for these economic settings and transfer functions are defined depending on agents' welfare weights. This set of mechanisms is compelling since interim efficient mechanisms are the best in the sense that there is no other incentive compatible and individually rational mechanism that is preferred by each agent.

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1 Introduction

We study the classical Bayesian environments with one-dimensional private signals, quasi-linear preferences, and interdependent valuations where individuals as a society have to choose an alternative from the set of feasible alternatives and decide how to arrange side payments to compensate all losers. Each agent obtains a private signal about his type and an agent's valuation for a given social alternative depends on her signal and on the other agents' one-dimensional signal (or information).

In these environments, a mechanism takes in the reported preferences of individuals and in turn picks an alternative (or a probability distribution over the set alternatives) and a vector of side transfers for every possible profile of message space. We restrict our attention to direct mechanisms in which message space is equal to type space for all individual. By revelation principle, there is no loss of generality in restricting our attention to such simple mechanisms.

We are interested in characterizing interim efficient mechanisms which are the *best* we can do given the incentive compatibility constraints (or strategy proofness). This set of mechanisms are important because it specifies all possible mechanisms that we might observe in practice and it also contains the optimal mechanism. In defining interim efficient mechanisms, we use the idea of interim incentive efficiency which was first

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introduced by Holmstrom and Myerson [4]. Incentive efficiency is a natural extension of efficiency to the Bayesian environments. A mechanism is incentive efficient if it is incentive compatible and there is no other incentive compatible mechanism which makes some types of individuals better off without hurting other types of individuals. There are three different stages to define incentive efficiency: ex ante stage, before any individual learns his type; interim stage, each individual knows his type but not the others' types; ex post stage, types are common knowledge.[1] In this paper, we concentrate on interim stage and characterize interim incentive efficient mechanisms with interdependent valuations.

Our characterization is general which can be applied to different economic settings such as nonlinear pricing by a monopolist, optimal income taxation, regulation of a monopolist, public good provision and (multi-object either with identical objects or heterogenous objects) auctions by redefining the set of feasible alternatives and the set of agents.[2]

Before presenting the model and results, let us discuss the relationship between our paper and other papers in additional detail. Our analysis is closely related to Holmstrom and Myerson [4], Cremer and McLean [1] and Ledyard and Palfrey [9], among others. Holmstrom and Myerson [4] consider the relationship among efficiency concepts and incentive efficiency concepts. Cremer and McLean [1] mainly focus on methods to extract surplus in multi-object auctions when types are correlated and they also allow for interdependent valuations. They provide sufficient conditions on information structures (a seller's subjective probability distribution over the set of possible types of agents) under which full surplus extraction is possible. We study a general social choice problem. In our setup, we assume that signals are independently drawn across agents and we provide explicit forms of the interim efficient mechanisms which also contains the optimal mechanism.[3] Ledyard and Palfrey [9] fully characterize interim efficient allocation rules for the class of independent linear environments with private valuations. We extend their characterization to the class of Bayesian environments with interdependent valuations.

[1]See, for example, Wilson [13] and Gresik [3] for characterization of ex ante incentive efficient mechanisms for bilateral trade and Ledyard and Palfrey [8] for characterization of interim incentive efficient mechanisms for public good environments. It is easy to show that the set of ex ante incentive efficient mechanisms is a subset of the set of interim incentive efficient mechanisms.

[2]The set of agents can be redefined by adding a seller and assigning initial property rights to the seller.

[3] The optimal mechanism can be found by equating the welfare weight of the seller to one and welfare weights of buyers to zero in auction design.

The present paper is a direct continuation of these papers.

The paper is organized as follows: In the next section, we present the environment and the basic notation. In Section 3 we reformulate the set of constraints and provide necessary and sufficient conditions for incentive compatibility and individual rationality. Then, we present the characterization result and proofs. In Section 4 we study regular problems whose solutions can be obtained by pointwise maximization and provide a sufficient condition for regularity. We make some concluding remarks in section 5.

2 The model

There are n individual agents. The set of agents is denoted by $N = \{1, ..., n\}$. Each agent privately observes his own type θ^i where each θ^i is independently drawn from cumulative distribution function $F^i(.)$ on $\Theta^i = [\underline{\theta}^i, \overline{\theta}^i]$ with $0 \leq \underline{\theta}^i \leq \overline{\theta}^i < \infty$. We assume $f^i(\theta^i) > 0$ for all θ^i. We denote a generic profile of agent valuations by $\theta = (\theta^1, ..., \theta^n) \in \Theta \equiv \times_{i=1}^N \Theta^i$. We use $\theta^{-i} = (\theta^1, ..., \theta^{i-1}, \theta^{i+1}, ..., \theta^n)$, and $\theta = (\theta^i, \theta^{-i})$ where $f(\theta) = \prod_{i=1}^N f^i(\theta^i)$. Let X be the set of possible decisions, or allocations (e.g., X could be a subset of an Euclidean space and represent the set of possible allocations of private and public goods).

The (ex-post) utility of agent i, $U^i : X \times \Theta \times \mathbb{R} \to \mathbb{R}$ depends not only on her own type, but also on the types of other agents as well. Moreover, we assume that utility function is quasi-linear on monetary transfers.

$$U^i(x^i, \theta, t^i) = v^i(x^i, \theta) - t^i(\theta),$$

where $v^i(x^i, ., \theta^{-i})$ is differentiable, monotone increasing and continuous for all i, x^i, θ^{-i}.

Let $\Delta(X)$ set of probability distribution over the set of possible allocations. A feasible mechanism $\zeta = (y, t)$ consists of a probabilistic allocation rule y and a payment function t where $y : \Theta \to \Delta(X)$ is a function from agents's reported type to a probability distribution over allocations such that $\sum_{x \in X} y^x(\theta) = 1$ and $y^x(\theta) \geq 0$ for all $\theta \in \Theta$. Let Y be the set of all possible allocation rules and $F \subseteq Y$ be the set of all feasible allocation rules. The payment function $t : \Theta \to \mathbb{R}^N$ is a map from the agents's reported type to a money payment by the agents where $\sum_{i=1}^N t^i(\theta) \geq 0$.[4]

[4]This condition requires that there is no outside source. Therefore a mechanism cannot run a deficit.

In the rest of the paper we only consider direct mechanisms in which the set of reported types is equal to the set of possible types. By the revelation principle, any allocation rule that results from equilibrium in any mechanism is also an equilibrium allocation rule of an incentive compatible, direct mechanism. Therefore, there is no loss of generality in restricting our attention to these simple type of mechanisms.

Let $U^i(\zeta, \theta^i, s^i)$ be the interim expected utility of agent i when he reports $s^i \neq \theta^i$, assuming all other agents truthfully report their type. That is

$$U^i(\zeta, \theta^i, s^i) = \int_{\Theta_{-i}} \left[\sum_{x \in X} y^x(s^i, \theta^{-i}) v^i(x^i, \theta) - t^i(s^i, \theta^{-i}) \right] dF^{-i}(\theta^{-i}).$$

Denote $U^i(\zeta, \theta^i) \equiv U^i(\zeta, \theta^i, \theta^i)$. The ex-ante utility of agent i is

$$U^i(\zeta) = \int_{\Theta} \left[\sum_{x \in X} y^x(\theta) v^i(x^i, \theta) - t^i(\theta) \right] dF^i(\theta).$$

Define also the conditional expected payment function $a^i : \Theta^i \to \mathbb{R}$ such that

$$a^i(\theta^i) = \int_{\Theta^{-i}} t^i(\theta) dF^{-i}(\theta^{-i}).$$

Following definitions which are standard in mechanism design describe our objective function and the constraint set.

Definition 1 *A mechanism ζ is interim incentive compatible if and only if* $U^i(\zeta, \theta^i) \geq U^i(\zeta, \theta^i, s^i)$ *for all* $i, \theta^i, s^i \in [\underline{\theta}^i, \overline{\theta}^i]$.

Definition 2 *A mechanism ζ satisfies interim individual rationality if and only if* $U^i(\zeta, \theta^i) \geq 0$ *for all* i, θ^i.

Definition 3 *A feasible mechanism ζ satisfies interim efficient if and only if (a) ζ is feasible (b) ζ is interim incentive compatible (c) ζ satisfies interim individual rationality and there is no other mechanism $\widehat{\zeta}$ such that (a) $\widehat{\zeta}$ is interim incentive compatible (b) $\widehat{\zeta}$ satisfies interim individual rationality (c)* $U^i(\widehat{\zeta}, \theta^i, \theta^i) \geq U^i(\zeta, \theta^i, \theta^i)$ *for all* i, θ^i *and* $U^i(\widehat{\zeta}, \theta^i, \theta^i) > U^i(\zeta, \theta^i, \theta^i)$ *for some i and for all* $\theta^i \in \widetilde{\Theta}^i \subset \Theta^i$*, where $\widetilde{\Theta}^i$ has strictly positive measure relative to* Θ^i.

Interim efficient mechanisms can be represented as the solutions to a set of maximization problems. The following well-known theorem is due to Holmstrom and Myerson [4].

Theorem 1 *A mechanism ζ is an interim efficient mechanism if and only if there exists $\lambda = \{\lambda^i : \Theta^i \to \mathbb{R}^+\}_{i=1}^N$ with $\int_{\underline{\theta}^i}^{\overline{\theta}^i} \lambda^i(\theta^i)dF^i(\theta^i) > 0$ for some i, such that ζ maximizes $\sum_{i=1}^N \int_{\underline{\theta}^i}^{\overline{\theta}^i} \lambda^i(\theta^i)U^i(\zeta(\theta),\theta^i)dF^i(\theta^i)$ subject to (a) ζ is feasible (b) ζ is interim incentive compatible (c) ζ satisfies interim individual rationality constraint.*

We now proceed to characterize the complete set of interim efficient mechanisms.

3 The Characterization

It is difficult to work with the constraint set in Theorem 1. In this section, we first start to reformulate the constraint set such that we can provide necessary and sufficient conditions for incentive compatibility and individual rationality. The second step in the characterization involves a general solution to the maximization problem with the constraints rewritten as described above.

3.1 Interim Incentive Compatible Mechanisms

This constraint requires that it is a Bayesian equilibrium for each agent to report her type truthfully. Let $S^i : \Theta^i \to \mathbb{R}$ be agent i's (expected) surplus function. Then given a mechanism ζ surplus function is

$$S^i(\theta^i) := sup\{U^i(\zeta, \theta^i, s^i)|s^i \in \Theta^i\}.$$

This optimization problem determines the agent i's optimal report. It is easy to see that a mechanism is (interim) incentive compatible if and only if $S^i(\theta^i) = U^i(\zeta, \theta^i)$ for all i, θ^i. Moreover, it is standard in mechanism design to characterize incentive compatibility by means of an envelope and a monotonicity condition. The following result which is useful in our characterization states that the derivative of the surplus function should be nondecreasing and the surplus function is uniquely determined by the expected surplus of the lowest type and the allocation rule. The proof is relatively standard, see for example Gresik [3], and Jehiel et al. [6].

Lemma 1 *A mechanism ζ is incentive compatible if and only if*

$$(s^i - \theta^i) \times (Q^i(s^i) - Q^i(\theta^i)) \geq 0 \quad \textit{for all } s^i, \theta^i \in \Theta^i \tag{1}$$

$$S^i(\theta^i) = S^i(\underline{\theta}^i) + \int_{\underline{\theta}^i}^{\theta^i} Q^i(s)ds \tag{2}$$

where

$$Q^i(\theta^i) \equiv \int_{\Theta^{-i}} \sum_{x \in X} y^x(\theta^i, \theta^{-i}) \frac{\partial v^i(x^i, \theta)}{\partial \theta^i} dF^{-i}(\theta^{-i}).$$

The first condition is the standard monotonicity condition and implies that $\frac{\partial S^i(\theta^i)}{\partial \theta^i} \geq 0$.[5] The second condition is the envelope condition.[6] The monotonicity condition have implications only on allocation rules. On the other hand, the envelope condition have implications on both allocation rules and payment functions. The above result also implies that

$$a^i(\theta^i) = \int_{\Theta^{-i}} \sum_{x \in X} y^x(\theta^i, \theta^{-i}) v^i(x^i, \theta) dF^{-i}(\theta^i) - S^i(\theta^i) \quad for\ all\ i \in N,$$

since for all incentive compatible mechanisms $S^i(\theta^i) = U^i(\zeta, \theta^i)$ for all i, θ^i. Notice that the expected payment function a is completely determined by $a(\underline{\theta}^i)$ and the allocation rule y since the constant of integration for all $i \in N$, $S^i(\underline{\theta}^i)$ is uniquely determined by $a(\underline{\theta}^i)$.

The main idea of the proof depends on showing that the following statements are equivalent: (1) ζ is incentive compatible for buyer i; (2) $S^i(\theta^i) = U^i(\zeta, \theta^i)$ for all $\theta^i \in \Theta^i$; (3) for all $\theta^i \in \Theta^i$, $Q^i(\theta^i) \in \partial S^i(\theta^i)$. A similar result is also stated by Rochet [12] for independent quasi-linear environments.

Now we can write agent i's (ex-ante) expected payment in an incentive compatible mechanism using the result above.

$$\begin{aligned}
\int_\Theta t^i(\theta) f(\theta) d\theta &= \int_{\Theta^i} a^i(\theta^i) dF(\theta^i) \\
&= \int_\Theta \sum_{x \in X} y^x(\theta) v^i(x^i, \theta) dF(\theta) - \int_{\Theta_i} S^i(\theta^i) dF^i(\theta^i)
\end{aligned}$$

$$= \int_\Theta \sum_{x \in X} y^x(\theta) v^i(x^i, \theta) dF(\theta) - S^i(\underline{\theta}^i) - \int_{\Theta^i} [\int_{\underline{\theta}^i}^{\theta^i} Q^i(s) ds] dF^i(\theta^i)$$

$$= \int_\Theta \sum_{x \in X} y^x(\theta) v^i(x^i, \theta) dF(\theta) - S^i(\underline{\theta}^i) - \int_{\Theta^i} Q^i(s) [\int_{s^i}^{\overline{\theta}^i} dF^i(\theta^i)] ds$$

[5] S^i is convex, continuous, and monotonically increasing. This implies S^i is differentiable almost everywhere and if it is differentiable at θ^i then $Q^i(\theta^i) = \partial S^i(\theta^i)$.

[6] See, for example, Milgrom and Segal [10].

$$= \int_{\Theta} \sum_{x \in X} y^x(\theta) \left(v^i(x^i, \theta) - \frac{\partial v^i(x^i, \theta)}{\partial \theta^i} \frac{1 - F^i(\theta^i)}{f^i(\theta^i)} \right) dF(\theta) - S^i(\underline{\theta}^i).$$

Then the expected budget surplus can be written as

$$B(\zeta) \equiv \sum_{i=1}^{N} \int_{\Theta} \sum_{x \in X} y^x(\theta) \left(v^i(x^i, \theta) - \frac{\partial v^i(x^i, \theta)}{\partial \theta^i} \frac{1 - F^i(\theta^i)}{f^i(\theta^i)} \right) dF(\theta) - \sum_{i=1}^{N} S^i(\underline{\theta}^i)$$

Notice that if ζ is resource feasible then $B(\zeta) \geq 0$ since there is no outside source.

In the next section we start to examine the implications of individual rationality constraints.

3.2 Interim Individually Rational Mechanisms

Individual rationality requires that each type of each agent must be at least as well off by participating as they would be by not participating at the interim stage. We assume that outside options are exogenously given and normalized to zero. Therefore, there is no consumption externality in our model. In other situations, it might be more appropriate to assume that outside options are type dependent or endogenous.[7]

Lemma 2 *An incentive compatible mechanism ζ is individually rational if and only if for all $i \in N$, $S^i(\underline{\theta}^i) \geq 0$.*

Proof: Individual rationality is satisfied if and only if $U^i(\zeta, \theta^i) \geq 0$ for all i, θ^i. By incentive compatibility

$$U^i(\zeta, \theta^i) = S^i(\theta^i) = S^i(\underline{\theta}^i) + \int_{\underline{\theta}^i}^{\theta^i} Q^i(s) ds \geq 0 \quad \textit{for all } i \in N, \theta^i \in \Theta^i$$

That is, it requires

$$min_{\theta^i \in \Theta^i} \left[S^i(\underline{\theta}^i) + \int_{\underline{\theta}^i}^{\theta^i} Q^i(s) ds \right] \geq 0,$$

$$\Leftrightarrow$$

[7] See Krishna and Perry [7], Jehiel et al. [6], Ledyard and Palfrey [9] and Figueroa and Skreta [2] for applications of type-dependent outside options in different economic settings.

$$S^i(\underline{\theta}^i) + min_{\theta^i \in \Theta^i}\left[\int_{\underline{\theta}^i}^{\theta^i} Q^i(s)ds\right] \geq 0 \Leftrightarrow S^i(\underline{\theta}^i) \geq 0 \quad for\ all\ i \in N.$$

The other direction is trivial since $v^i(x^i, ., \theta^{-i})$ is monotone increasing. ∎

When we combine incentive compatibility and individual rationality, we get

$$\sum_{i=1}^{N}\int_{\Theta}\sum_{x\in X} y^x(\theta)\left(v^i(x^i,\theta) - \frac{\partial v^i(x^i,\theta)}{\partial\theta^i}\frac{1-F^i(\theta^i)}{f^i(\theta^i)}\right)dF(\theta) \geq \sum_{i=1}^{N} S^i(\underline{\theta}^i) \geq 0$$

since $S^i(\underline{\theta}^i) \geq 0$ for all $i \in N$ from individual rationality.

3.3 Interim Efficient Mechanisms

Before stating the main characterization, the following definition and lemma will be useful in reformulating the original problem.

Definition 4 *If* $\lambda^{0i} \equiv \int_{\underline{\theta}^i}^{\overline{\theta}^i} \lambda^i(\theta^i)dF^i(\theta^i) > 0$, *let* $\Lambda^i(\theta^i) = \frac{1}{\lambda^{0i}}\int_{\underline{\theta}^i}^{\theta^i} \lambda^i(s)dF^i(s)$. *If* $\lambda^{0i} = 0$, *let* $\Lambda^i(\theta^i) = 0$.

λ^{0i} is agent i's ex ante welfare weight relative to other agents. $\Lambda^i(\theta^i)$ is a relative weight of agent i's lower types.

Lemma 3

$$\int_{\underline{\theta}^i}^{\overline{\theta}^i} \lambda^i(\theta^i)\left(S^i(\underline{\theta}^i) + \int_{\underline{\theta}^i}^{\theta^i} Q^i(s)ds\right)dF^i(\theta^i)$$

$$=$$

$$\lambda^{0i}\left[S^i(\underline{\theta}^i) + \int_{\underline{\theta}^i}^{\overline{\theta}^i}(\frac{1-\Lambda^i(\theta^i)}{f^i(\theta^i)})Q^i(\theta^i)dF^i(\theta^i)\right].$$

Proof: By changing the order of integration we get:

$$\begin{aligned}
LHS &= \lambda^{0i}S^i(\underline{\theta}^i) + \int_{\underline{\theta}^i}^{\overline{\theta}^i} Q^i(s)\left[\int_{s}^{\overline{\theta}^i}\lambda^i(\theta^i)dF^i(\theta^i)\right]ds \\
&= \lambda^{0i}\left[S^i(\underline{\theta}^i) + \int_{\underline{\theta}^i}^{\overline{\theta}^i} Q^i(s)(1-\Lambda^i(s))ds\right] \\
&= \lambda^{0i}\left[S^i(\underline{\theta}^i) + \int_{\underline{\theta}^i}^{\overline{\theta}^i}(\frac{1-\Lambda^i(\theta^i)}{f^i(\theta^i)})Q^i(\theta^i)dF^i(\theta^i)\right].
\end{aligned}$$

∎

We can now provide a characterization for the reformulated problem. This result is much more easier to work with than the result in Theorem 1.

Theorem 2 *A mechanism $\zeta = (y,t)$ is interim efficient if and only if there exists non-negative type-dependent welfare weights, $\{\lambda^i\}_{i=1}^N$, where $\sum_{i\in N}\lambda^{0i} > 0$, and N constants, $\{a^i(\underline{\theta}^i)\}_{i=1}^N$, such that $(y, \{a^i(\underline{\theta}^i)\}_{i=1}^N)$ solves,*

$$max_{y\in F}\sum_{i=1}^N \lambda^{0i}\left[S^i(\underline{\theta}^i) + \int_{\underline{\theta}^i}^{\overline{\theta}^i}(\frac{1-\Lambda^i(\theta^i)}{f^i(\theta^i)})Q^i(\theta^i)dF^i(\theta^i)\right] \tag{3}$$

subject to

$$\sum_{i=1}^N\int_\Theta\sum_{x\in X}y^x(\theta)\left(v^i(x^i,\theta) - \frac{\partial v^i(x^i,\theta)}{\partial\theta^i}\frac{1-F^i(\theta^i)}{f^i(\theta^i)}\right)dF(\theta) - \sum_{i=1}^N S^i(\underline{\theta}^i) \geq 0 \tag{4}$$

$$S^i(\underline{\theta}^i) = \int_{\Theta^{-i}}\sum_{x\in X}y^x(\underline{\theta}^i,\theta^{-i})v^i(x^i,\theta)dF^{-i}(\theta^{-i}) - a^i(\underline{\theta}^i) \geq 0 \quad for\ all\ i \tag{5}$$

$$Q^i(\theta^i) \quad monotone\ increasing\ for\ all\ i,\theta^i \tag{6}$$

where the payment function is given by:

$$a^i(\theta^i) = \int_{\Theta^{-i}}t^i(\theta)dF^{-i}(\theta^{-i}) = a^i(\underline{\theta}^i) - \int_{\underline{\theta}^i}^{\theta^i}Q^i(s)ds. \tag{7}$$

Proof: Follows from Lemmas 1, 2, 3 and the discussion above. (4) is resource feasibility, (5) is individual rationality, and (6) is incentive compatibility. The expected payment function is given by

$$\begin{aligned}
a^i(\theta^i) &= \int_{\Theta^{-i}}\sum_{x\in X}y^x(\theta^i,\theta^{-i})v^i(x^i,\theta)dF^{-i}(\theta^i) - S^i(\theta^i) \\
&= \int_{\Theta^{-i}}\sum_{x\in X}y^x(\theta^i,\theta^{-i})v^i(x^i,\theta)dF^{-i}(\theta^i) - S^i(\underline{\theta}^i) - \int_{\underline{\theta}^i}^{\theta^i}Q^i(s)ds \\
&= a^i(\underline{\theta}^i) - \int_{\underline{\theta}^i}^{\theta^i}Q^i(s)ds.
\end{aligned}$$

∎

4 Regular Problems

Following the same idea in Myerson [11] we characterize the solution to the problem in Theorem 2 for the case where monotonicity constraint is not binding. In this case solution can be obtained by pointwise maximizing the integrand in the objective function. Then we provide conditions under which the solutions to this reduced problem satisfies the monotonicity constraint. When the solutions to the original problem and reduced problem coincide, we refer to problem as regular.

Given non-negative type-dependent welfare weights, $\{\lambda^i\}_{i=1}^N$, we can define the Lagrangian function as

$$\mathcal{L}(y,(a^i(\underline{\theta}^i))_{i=1}^N,\gamma,(\mu^i)_{i=1}^N)=\sum_{i=1}^N \lambda^{0i}\left[S^i(\underline{\theta}^i)+\int_{\underline{\theta}^i}^{\overline{\theta}^i}(\frac{1-\Lambda^i(\theta^i)}{f^i(\theta^i)})Q^i(\theta^i)dF^i(\theta^i)\right]$$

$$+\gamma\left[\sum_{i=1}^N\int_\Theta\sum_{x\in X}y^x(\theta)\left(v^i(x^i,\theta)-\frac{\partial v^i(x^i,\theta)}{\partial\theta^i}\frac{1-F^i(\theta^i)}{f^i(\theta^i)}\right)dF(\theta)-\sum_{i=1}^N S^i(\underline{\theta}^i)\right]$$

$$+\sum_{i=1}^N\mu^i\left(\int_{\Theta^{-i}}\sum_{x\in X}y^x(\underline{\theta}^i,\theta^{-i})v^i(x^i,\theta)dF^{-i}(\theta^{-i})-a^i(\underline{\theta}^i)\right)$$

The first order conditions with respect to γ (resource feasibility multiplier) and with respect to μ^i (individual rationality multiplier) imply that

$$\gamma\geq 0, B(y)\geq 0 \;\; and \;\; \gamma B(y)=0,$$

$$\mu^i\geq 0, S^i(\underline{\theta}^i)\geq 0 \;\; and \;\; \mu^i S^i(\underline{\theta}^i)=0 \;\; for \ all \ i\in N.$$

The first order condition with respect to $a^i(\underline{\theta}^i)$ yields $-\lambda^{0i}+\gamma-\mu^i=0$. Then $\gamma\geq\lambda^{0i}$ for all $i\in N$. This implies the budget surplus constraint is always binding ($\gamma>0$) since there is $i\in N$ such that $\lambda^{0i}>0$ and $\mu^i\succeq 0$ for all $i\in N$. Let $\overline{\lambda}=max_{i\in N}\{\lambda^{0i}\}$. There are two possible cases:

Case 1: For all $i\in N$, $\gamma>\overline{\lambda}\Rightarrow\mu^i>0\Rightarrow S^i(\underline{\theta}^i)=0$.

Case 2: For each $k\in K$, $\gamma=\overline{\lambda}=\lambda^{0k}\Rightarrow\mu^k=0\Rightarrow S^k(\underline{\theta}^k)\geq 0$ and for each $m\in M$, $\gamma=\overline{\lambda}>\lambda^{0m}\Rightarrow\mu^m>0\Rightarrow S^m(\underline{\theta}^m)=0$, where $N=K\cup M$.

The following Lemma summarizes the discussion above.

Lemma 4

$$\begin{aligned}\sum_{i=1}^{N}\lambda^{0i}S^i(\underline{\theta}^i) &= \overline{\lambda}\sum_{i=1}^{N}S^i(\underline{\theta}^i)\\ &= \overline{\lambda}\sum_{i=1}^{N}\left[\int_{\Theta}\sum_{x\in X}y^x(\theta)v^i(x^i,\theta)dF(\Theta)+\int_{\Theta^i}\frac{1-F^i(\theta^i)}{f^i(\theta^i)}Q^i(\theta^i)dF^i(\theta^i)\right].\end{aligned}$$

Proof: From Case 1 and 2, if $S^i(\underline{\theta}^i)\neq 0$ for some $i\in N$ then their ex ante welfare weights are equal to $\overline{\lambda}$. The second equality follows by rearranging the terms in resource feasibility constraint which is always binding. ∎

Using the above result the objective function can be written as follows[8]

$$\begin{aligned}\sum_{i=1}^{N}\Bigg[\int_{\Theta}\sum_{x\in X}y^x(\theta)\left(v^i(x^i,\theta)-\frac{\partial v^i(x^i,\theta)}{\partial\theta^i}\frac{1-F^i(\theta^i)}{f^i(\theta^i)}\right)dF(\theta)\\ +\frac{\lambda^{0i}}{\overline{\lambda}}\int_{\Theta^i}(\frac{1-\Lambda^i(\theta^i)}{f^i(\theta^i)})Q^i(\theta^i)dF^i(\theta^i)\Bigg]=\end{aligned}$$

$$\sum_{i=1}^{N}\int_{\Theta}\sum_{x\in X}y^x(\theta)\left[v^i(x^i,\theta)+\frac{\partial v^i(x^i,\theta)}{\partial\theta^i}\left(\frac{F^i(\theta^i)-1}{f^i(\theta^i)}+\frac{\lambda^{0i}}{\overline{\lambda}}\frac{1-\Lambda^i(\theta^i)}{f^i(\theta^i)}\right)\right]dF(\theta)$$

Suppose we are in Case 1. In this case both constraints are binding. This implies, $\sum_{i=1}^{N}\int_{\Theta}\sum_{x\in X}y^x(\theta)\left(v^i(x^i,\theta)-\frac{\partial v^i(x^i,\theta)}{\partial\theta^i}\frac{1-F^i(\theta^i)}{f^i(\theta^i)}\right)dF(\theta)=0$ from resource feasibility and $\int_{\Theta^{-i}}\sum_{x\in X}y^x(\underline{\theta}^i,\theta^{-i})v^i(x^i,\theta)dF^{-i}(\theta^{-i})=a^i(\underline{\theta}^i)$ from individual rationality. Therefore, we can uniquely solve for the set of expected payments of all agents' minimum types.

Suppose now we are in Case 2. The argument for each $i\in M$ is similar to Case 1. On the other hand, for each $i\in K$ the individual rationality constraint is not binding. Therefore, we need K constants to solve for the payment function. Note that the agents with the highest ex ante welfare weight shares the surplus to make the resource constraint binding.

When we combine both cases and rearrange terms, we get the following result. It is much more easier to work with than Theorem 2 since we have less number of constant to work with unless every agent has same ex ante welfare weights.

[8] We can divide the objective function by $\overline{\lambda}$ since it is exogenous.

Theorem 3 *For regular problems $\zeta = (y,t)$ is interim efficient if and only if there exist non-negative type-dependent welfare weights, $\{\lambda^i\}_{i=1}^N$, where $\sum_{i\in N}\lambda^{0i} > 0$, $\gamma \geq \overline{\lambda}$, and $K \leq N$ constants, $\{a^k(\underline{\theta}^k)|\lambda^{0k} = \gamma = \overline{\lambda}, \forall\ k \in K\}$, such that $y \in \Delta(X)$ simultaneously solves the following inequalities* [(8), (9), (10)];

$$\max_{y\in F} \sum_{i=1}^N \int_\Theta \sum_{x\in X} y^x(\theta)\Big[v^i(x^i,\theta) + \frac{\partial v^i(x^i,\theta)}{\partial\theta^i}\left(\frac{F^i(\theta^i)-1}{f^i(\theta^i)} + \frac{\lambda^{0i}}{\gamma}\frac{1-\Lambda^i(\theta^i)}{f^i(\theta^i)}\right)\Big]dF(\theta) \tag{8}$$

$$0 \leq \sum_{i=1}^N \int_\Theta \sum_{x\in X} y^x(\theta)\left(v^i(x^i,\theta) - \frac{\partial v^i(x^i,\theta)}{\partial\theta^i}\frac{1-F^i(\theta^i)}{f^i(\theta^i)}\right)dF(\theta) \tag{9}$$

$$0 = (\gamma - \overline{\lambda})\left[\sum_{i=1}^N \int_\Theta \sum_{x\in X} y^x(\theta)\left(v^i(x^i,\theta) - \frac{\partial v^i(x^i,\theta)}{\partial\theta^i}\frac{1-F^i(\theta^i)}{f^i(\theta^i)}\right)dF(\theta)\right] \tag{10}$$

and the payment function is given by:

$$\forall\ i \in M,\quad a^i(\theta^i) = \int_{\Theta^{-i}} \sum_{x\in X} y^x(\theta)v^i(x^i,\theta)dF^{-i}(\theta^{-i}) - \int_{\underline{\theta}^i}^{\theta^i} Q^i(s)ds\ , \tag{11}$$

$$\forall\ i \in K,\quad a^i(\theta^i) = a^i(\underline{\theta}^i) - \int_{\underline{\theta}^i}^{\theta^i} Q^i(s)ds \tag{12}$$

where

$$0 \leq \int_{\Theta^{-i}} \sum_{x\in X} y^x(\underline{\theta}^i,\theta^{-i})v^i(x^i,\theta)dF^{-i}(\theta^{-i}) - a^i(\underline{\theta}^i)\quad \forall\quad i \in K, \quad \textit{and}$$

$$\begin{aligned}\sum_{i\in K} a^i(\underline{\theta}^i) \quad &= \quad \sum_{i\in K}\int_{\Theta^{-i}} \sum_{x\in X} y^x(\theta)v^i(x^i,\theta)dF^{-i}(\theta^{-i}) \\ &- \quad \sum_{i=1}^N \int_\Theta \sum_{x\in X} y^x(\theta)\left(v^i(x^i,\theta) - \frac{\partial v^i(x^i,\theta)}{\partial\theta^i}\frac{1-F^i(\theta^i)}{f^i(\theta^i)}\right)dF(\theta).\end{aligned}$$

Proof: It follows from Lemma 4 and the discussion for Case 1 and 2 as stated above. Note that

$$\sum_{i=1}^N S^i(\underline{\theta}^i) = \sum_{i\in K} S^i(\underline{\theta}^i) = \sum_{i\in K}\int_{\Theta^{-i}} \sum_{x\in X} y^x(\theta)v^i(x^i,\theta)dF^{-i}(\theta^{-i}) - \sum_{i\in K} a^i(\theta^i).$$

Then the constraint for the payment functions follows since resource feasibility constraint is binding. ∎

The set of integration constants $\{a^i(\underline{\theta}^i)_{i=1}^N\}$ may not be a singleton but the sum of integration constants $\sum_{i=1}^N a^i(\underline{\theta}^i)$ is uniquely determined by an interim efficient allocation rule y. Therefore, there might be different payment functions that implement the same allocation rule (i.e., different mechanisms with a same allocation rule).

The following result is a special case of Theorem 3 in which there is only one agent whose ex ante welfare weight is higher than the welfare weights of the other agents.

Corollary 1 *Suppose the type-dependent welfare weights are such that $\lambda^{0i} = \overline{\lambda} > \lambda^{0k}$ for all $k \in N\backslash\{i\}$ and $\gamma \geq \overline{\lambda}$. Then, for regular problems, a mechanism ζ is interim efficient if and only if $y \in \Delta(X)$ simultaneously solves the following inequalities*

$$\max_{y\in F} \sum_{i=1}^N \int_\Theta \sum_{x\in X} y^x(\theta)\Big[v^i(x^i,\theta) + \frac{\partial v^i(x^i,\theta)}{\partial\theta^i}\left(\frac{F^i(\theta^i)-1}{f^i(\theta^i)} + \frac{\lambda^{0i}}{\gamma}\frac{1-\Lambda^i(\theta^i)}{f^i(\theta^i)}\right)\Big]dF(\theta)$$

$$0 \leq \sum_{i=1}^N \int_\Theta \sum_{x\in X} y^x(\theta)\left(v^i(x^i,\theta) - \frac{\partial v^i(x^i,\theta)}{\partial\theta^i}\frac{1-F^i(\theta^i)}{f^i(\theta^i)}\right)dF(\theta)$$

$$0 = (\gamma - \overline{\lambda})\left[\sum_{i=1}^N \int_\Theta \sum_{x\in X} y^x(\theta)\left(v^i(x^i,\theta) - \frac{\partial v^i(x^i,\theta)}{\partial\theta^i}\frac{1-F^i(\theta^i)}{f^i(\theta^i)}\right)dF(\theta)\right]$$

where the payment function is given by:

$$\forall\ k \neq i, \quad a^k(\theta^k) = \int_{\Theta^{-k}} \sum_{x\in X} y^x(\theta)v^k(x^k,\theta)dF^{-k}(\theta^{-k}) - \int_{\underline{\theta}^k}^{\theta^k} Q^k(s)ds\ ,$$

$$\begin{aligned} a^i(\theta^i) = &\int_{\Theta^{-i}} \sum_{x\in X} y^x(\theta)v^i(x^i,\theta)dF^{-i}(\theta^{-i}) \\ &- \sum_{i=1}^N \int_\Theta \sum_{x\in X} y^x(\theta)\left(v^i(x^i,\theta) - \frac{\partial v^i(x^i,\theta)}{\partial\theta^i}\frac{1-F^i(\theta^i)}{f^i(\theta^i)}\right)dF(\theta) \\ &- \int_{\underline{\theta}^i}^{\theta^i} Q^i(s)ds. \end{aligned}$$

Proof: Finding the allocation rule and the payment functions for all $k \neq i$ directly follows from Theorem 3. Since individual rationality constraint is binding for all $-i$, $\sum_{j=1}^N S^j(\underline{\theta}^j) = S^i(\underline{\theta}^i) = \int_{\Theta^{-i}} \sum_{x\in X} y^x(\underline{\theta}^i,\theta^{-i})v^i(x^i,\theta)dF^{-i}(\theta^{-i})$
$-\ a^i(\underline{\theta}^i)$. We know that resource feasibility constraint is always binding. This implies agent i receives the remaining surplus. ∎

4.1 Regular Problems without Individual Rationality

It is much more easier to solve the problem without individual rationality. First order conditions imply $\lambda^{0i} = \gamma$ for all $i \in N$. The ex-ante welfare weights must all be equal. Otherwise, the solution does not exist since it is always possible to improve welfare by making arbitrarily large transfers between agents with different welfare weights. Then the resource constraint is always binding.

Corollary 2 *A mechanism $\zeta = (y, t)$ is interim efficient if and only if there exists N constants, $\{a^i(\underline{\theta}^i)\}_{i=1}^N$, such that the allocation rule $y \in \Delta(X)$ solves,*

$$\underset{y \in F}{max} \sum_{i=1}^{N} \int_{\Theta} \sum_{x \in X} y^x(\theta) \Big[v^i(x^i, \theta) + \frac{\partial v^i(x^i, \theta)}{\partial \theta^i} \left(\frac{F^i(\theta^i) - \Lambda^i(\theta^i)}{f^i(\theta^i)} \right) \Big] dF(\theta)$$

and the payment function is given by:

$$a^i(\theta^i) = \int_{\Theta^{-i}} t^i(\theta) dF^{-i}(\theta^{-i}) = a^i(\underline{\theta}^i) - \int_{\underline{\theta}^i}^{\theta^i} Q^i(s) ds$$

where

$$\begin{aligned} \sum_{i=1}^{N} a^i(\underline{\theta}^i) &= \sum_{i=1}^{N} \Bigg[\int_{\Theta^{-i}} \sum_{x \in X} y^x(\theta) v^i(x^i, \theta) dF^{-i}(\theta^{-i}) \\ &\quad - \int_{\Theta} \sum_{x \in X} y^x(\theta) \left(v^i(x^i, \theta) - \frac{\partial v^i(x^i, \theta)}{\partial \theta^i} \frac{1 - F^i(\theta^i)}{f^i(\theta^i)} \right) dF(\theta) \Bigg]. \end{aligned}$$

Proof: The argument above implies that $\lambda^{0i} = \gamma$ for all $i \in N$ and we know that resource feasibility constraint is always binding. From Theorem 2, the objective function can be written as follows

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\Theta} \sum_{x \in X} y^x(\theta) \Big[v^i(x^i, \theta) \\ &\qquad + \frac{\partial v^i(x^i, \theta)}{\partial \theta^i} \left(\frac{F^i(\theta^i) - 1}{f^i(\theta^i)} + \frac{\lambda^{0i}}{\gamma} \frac{1 - \Lambda^i(\theta^i)}{f^i(\theta^i)} \right) \Big] dF(\theta) \\ &= \sum_{i=1}^{N} \int_{\Theta} \sum_{x \in X} y^x(\theta) \Big[v^i(x^i, \theta) + \frac{\partial v^i(x^i, \theta)}{\partial \theta^i} \left(\frac{F^i(\theta^i) - \Lambda^i(\theta^i)}{f^i(\theta^i)} \right) \Big] dF(\theta) \end{aligned}$$

The payment function directly follows from Theorem 2 and the constraint on some of expected payments follows from resource feasibility. ∎

Corollary 2 implies that if there is no interim individual rationality constraints, an interim efficient mechanism can divide the surplus arbitrarily among agents as long as the condition on the sum of expected payments is satisfied.

4.2 A Sufficient Condition for Regularity

In this section we provide a sufficient condition under which the solution to the regular problem coincide with the solution to the original problem in Theorem 2. Let

$$W^i(\theta, x, \lambda^0) = v^i(x^i, \theta) + \frac{\partial v^i(x^i, \theta)}{\partial \theta^i}\left(\frac{F^i(\theta^i) - 1}{f^i(\theta^i)} + \frac{\lambda^{0i}}{\overline{\lambda}}\frac{1 - \Lambda^i(\theta^i)}{f^i(\theta^i)}\right). \tag{13}$$

We can call $W^i(\theta, x, \lambda^0)$ as the virtual valuation of agent i for allocation x following Myerson [11]. Substituting (13) into (8) gives us:

$$\max_{y \in F} \sum_{i=1}^{N} \int_{\Theta} \sum_{x \in X} y^x(\theta) W^i(\theta, x, \lambda^0) dF(\theta) \tag{14}$$

So the regular problem to find interim efficient mechanisms can be stated as (y, t) is an interim efficient mechanism if and only if the allocation rule $y \in \Delta(X)$ simultaneously solves (14), (9), (10) and the payment function t satisfies the conditions (11) and (12).

The problem stated above has a simple solution defined by[9]

$$y^x(\theta, \lambda) = \begin{cases} 1 & \text{if } x = argmax_{a \in X} \sum_{i=1}^{N} W^i(\theta, a, \lambda^0) \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

This implies an interim efficient mechanism assigns probability one to an allocation with the highest sum of virtual valuations.

We now provide a condition under which the solution (15) and the condition implies that the monotonicity constraint is satisfied. The condition is similar to the conditions in the environments of Jehiel and Moldovanu [5] and Figueroa and Skreta [2].

Assumption 4 *For all $i \in N$, all $\theta^i, s^i \in \Theta^i$, and all $\theta^{-i} \in \Theta^{-i}$ such that $\theta^i \geq s^i$;*

$$\partial_2 v^i(x^i, \theta^i, \theta^{-i}) \geq \partial_2 v^i(y^i, s^i, \theta^{-i})^{10}$$

[9] For simplicity, we assume that there are no allocations $x, y, x \neq y$ such that $\sum_{i=1}^{N} W^i(\theta, x, \lambda^0) = \sum_{i=1}^{N} W^i(\theta, y, \lambda^0)$. We can also use a random tie-breaking rule.

[10] $\partial_2 v^i(x^i, \theta^i, \theta^{-i}) = \frac{\partial v^i(x^i, \theta)}{\partial \theta^i}$.

where

$$x = \underset{a \in X}{argmax} \sum_{i=1}^{N} W^i(\theta, a, \lambda^0) \quad and \quad y = \underset{a \in X}{argmax} \sum_{i=1}^{N} W^i(s^i, \theta^{-i}, a, \lambda^0).$$

Assumption 1 basically restricts the set of admissible valuation functions such that interim efficient allocations rules satisfy the condition (1). It states that the derivative of the functions $v^i(.)$ is increasing for all interim efficient allocations. Note that the allocation is endogenous and depends on the vector of types. Let θ and s be two different vector of types. We already know by the initial assumption that $v^i(x^i, \theta) \geq v^i(x^i, s)$ if $\theta^i \geq s^i$. However, the allocation might also change and we should guarantee that the valuation functions of each agent is also increasing in this case.

Proposition 1 *If each $v^i(.)$ satisfies the Assumption 1, then the solution (15) satisfies all constraints in Theorem 2.*

Proof: By Assumption 1, $\partial_2 v^i(x^i, \theta)$ increases (decreases) when θ^i increases (decreases). Note that the allocation of agent i might also change when the agent has a different signal. Suppose $x = argmax_{a \in X} \sum_{i=1}^{N} W^i(\theta, a, \lambda^0)$, $y = argmax_{a \in X} \sum_{i=1}^{N} W^i(s^i, \theta^{-i}, a, \lambda^0)$, and $\theta^i \geq s^i$. Then,

$$Q^i(\theta^i) = \int_{\Theta^{-i}} \sum_{x \in X} y^x(\theta^i, \theta^{-i}) \frac{\partial v^i(x^i, \theta)}{\partial \theta^i} dF^{-i}(\theta^{-i}) = \int_{\Theta^{-i}} \frac{\partial v^i(x^i, \theta)}{\partial \theta^i} dF^{-i}(\theta^{-i})$$

$$\geq Q^i(s^i) = \int_{\Theta^{-i}} \frac{\partial v^i(y^i, s^i, \theta^{-i})}{\partial s^i} dF^{-i}(\theta^{-i}).$$

This implies $Q^i(\theta^i)$ is monotone increasing. Hence the solution (15) satisfies all constraints in Theorem 2. In case $s^i > \theta^i$ the same conclusion is obtained by reversing the inequalities. ∎

5 Concluding Remarks

In this paper, we have characterized interim efficient mechanisms for the classical Bayesian environments with one-dimensional private signals, quasi-linear preferences, and interdependent valuations. We showed that interim efficient allocation rules assigns probability one to an allocation that maximizes sum of agents' virtual valuations that are carefully defined for this environment. Moreover, we showed that the allocation rules can be implemented with different payment functions as long as the sum of payments is equal to the endogenously defined variable. We mostly concentrated on regular problems where we assumed that monotonicity constraint is not binding and provide a sufficient condition for regular problems. The extension of characterization to irregular problems remains open.

The extension of our characterization to the irregular problems and to the environments with multidimensional signal with or without interdependent valuations will be a subject of our future work.

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